Formula Sheet

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1 Kinematics

Velocity-distance relation under constant acceleration. Given an initial velocity v_0 and a distance d:

$$v^2 = {v_0}^2 + 2ad$$

Projectile motion distance:

$$d = \frac{v^2 \sin 2\theta}{g}$$

Center of mass:

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \sum_i \mathbf{r}_i m_i = \frac{1}{M} \int \mathbf{r} dm$$

2 Relativity

2.1 Kinematics

Let A be the "fixed" observer and B an observer moving with velocity v relative to A.

Loss of simultaneity. For two clocks a distance L apart in A's frame that are reading the same time in that frame, the "rear" clock from B's point of view will be faster by a factor of:

$$\Delta t = \frac{Lv}{c^2}$$

Beta and gamma factors:

$$\beta = \frac{v}{c},\ \gamma = \frac{1}{\sqrt{1-\beta^2}},\ \gamma > 1$$

Time dilation and length contraction:

$$t_B = \gamma t_A;\ L_B = \frac{L_A}{\gamma}$$

Velocity addition. Say A observes a motion of velocity v_A, then the velocity with respect to B is:

$$v_B = \frac{v_A + v}{1 + v v_A / c^2}$$

Lorentz transformations. Given that A has a coordinate system of (x, y, z, t), the coordinate system for B is (x', y, z, t') where:

$$\Delta x = \gamma(\Delta x' + v\Delta t'),\ \Delta t = \gamma \left(\Delta t' + \frac{v}{c^2}\Delta x'\right)$$

Time-space invariant. Given, for two events:

$$\Delta s^2 \equiv c^2 \Delta t^2 - \Delta x^2$$

The value Δs^2 is the same in any frame of reference.

2.2 Dynamics

We are given an observer and some system of mass m moving at a speed v.

Momentum and energy:

$$\mathbf{p} = \gamma m \mathbf{v},\ E = \gamma m c^2$$

Energy-momentum relations:

$$m^2 c^4 = E^2 - p^2 c^2,\ \frac{p}{E} = \frac{v}{c^2}$$

Energy/momentum for photons:

$$E = pc$$

Lorentz transformations for energy. Given a frame of reference moving with speed v, that measures for a system E' and p', we find in the nonmoving frame:

$$E = \gamma(E' + vp'),\ p = \gamma \left(p' + \frac{v}{c^2} E'\right)$$

Relativistic force:

$$F = \gamma^3 m a = \frac{dp}{dt} = \frac{dE}{dx}$$

3 Rotational Motion

Rolling without slipping:

$$\alpha = \frac{a}{r},\ \omega = \frac{v}{r}$$

Moment of inertia:

$$I = \sum m_i {r_i}^2 = \int r^2 dm$$

Parallel axis theorem. Given an axis of rotation parallel to an axis through the center of mass:

$$I_p = I_{\text{CM}} + M d^2$$

Perpendicular axis theorem. Given a *planar* object, with the $\mathbf{z}$ axis normal to it:

$$I_z = I_x + I_y$$

Definition of torque:

$$\vec{\tau} = \mathbf{r} \times \mathbf{F};\ \tau = rF\sin\theta$$

Torque and angular acceleration:

$$\sum \tau = I\alpha$$

Definition of angular momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \int (\mathbf{r} \times \mathbf{v})dm;\ L = rp\sin\theta$$

Torque and angular momentum. Given a center of rotation that is fixed either in an inertial frame or on the center of mass:

$$\vec{\tau} = \frac{d\mathbf{L}}{dt}$$

Angular momentum and velocity. For a system that is only rotating about a single axis:

$$L = I\omega$$

Angular impulse. In a system where a force is applied at a constant distance r from the point of rotation:

$$\Delta L = r\Delta p$$

Translation and rotation. Given an object with angular momentum $\mathbf{L}'$ about its own center of mass, the angular momentum about any other center is:

$$L = M\mathbf{R}_{\text{CM}} \times \mathbf{v}_{\text{CM}} + \mathbf{L}'$$

4 Harmonic Motion

Given a differential equation of the form $y'' = -\omega^2 y$, the solution will be:

$$y = A\cos(\omega t + \phi)$$

The constant ω is the angular frequency. The period T and frequency ν are:

$$T = \frac{2\pi}{\omega},\ \nu = \frac{1}{T} = \frac{\omega}{2\pi}$$

For a spring, $\omega = \sqrt{k/m}$; for a pendulum, $\omega = \sqrt{g/l}$. For a physical pendulum with moment of inertia *at the pivot* a distance d from the CM, $\omega = \sqrt{mgd/I}$.

Damped motion. Consider a damping force $F = -bv$ and a harmonic force $F = kx$. Then define:

$$\omega_0{}^2 = \frac{k}{m},\ \gamma = \frac{b}{2m};\ \omega'^2 = \omega_0{}^2 - \gamma^2,\ \Omega^2 = \gamma^2 - \omega_0{}^2$$

There are three possible cases:

Under	$\gamma < \omega_0$	$x(t) = Ae^{-\gamma t}\cos(\omega' t + \phi)$
Over	$\gamma > \omega_0$	$x(t) = Ae^{-(\gamma+\Omega)t} + Be^{-(\gamma-\Omega)t}$
Critical	$\gamma = \omega_0$	$x(t) = e^{-\gamma t}(A + Bt)$

Driven oscillation. In addition to the damping $-bv$ and harmonic kx forces, consider a driving force $F_d(t) = F\cos\omega_d t$:

$$x(t) = \frac{F}{mR}\cos(\omega_d t - \phi)$$

with the following constants:

$$R^2 = \left(\omega_0{}^2 - \omega_d{}^2\right)^2 + \left(2\gamma\omega_d\right)^2,\ \tan\phi = \frac{2\gamma\omega_d}{\omega_0{}^2 - \omega_d{}^2}$$

5 Universal Gravitation

Newton's Law of gravitation:

$$F = -\frac{Gm_1m_2}{r^2} \text{ where } G = 6.6726 \times 10^{-11}\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}$$

The units of G can also be $\text{m}^2\text{kg}^{-1}\text{s}^{-2}$.

Gravitational potential:

$$U(r) = -\frac{Gm_1m_2}{r}$$

Kepler's Laws. The planets move in elliptical orbits, they sweep out equal areas over equal times, and for an orbit with semimajor axis a and period T:

$$T^2 = \frac{4\pi^2a^3}{GM_{\text{sun}}}$$

6 Fictitious Forces

In an accelerated reference frame $\mathbf{R}$ with rotation $\vec{\omega}$, the force on an object is the sum of the real forces on it and the following "fictitious forces":

$$\begin{aligned} \text{Translational:}&\ -m\frac{d^2\mathbf{R}}{dt^2} \\ \text{Centrifugal:}&\ -m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ \text{Coriolis:}&\ -2m\vec{\omega} \times \mathbf{v} \\ \text{Azimuthal:}&\ -m\frac{d\vec{\omega}}{dt} \times \mathbf{r} \end{aligned}$$